Ch. 3 Second-Order Linear Differential Equations

§3.1 Homogeneous DEs with constant coefficients

Consider the 2nd-order DE of the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

which is linear if

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y$$

where g, p, and q are given functions of t.
Our general 2nd-order linear DE is

$$y'' + p(t)y' + q(t)y = g(t).$$

The initial value problem requires us to specify

$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y_0'$$

We now specify an initial point (t_0, y_0) and an initial slope y_0'.

The second order equation is called homogeneous if $g(t) = 0$ for all t. Otherwise, it is called nonhomogeneous.

Note: Solving the nonhomogeneous problem requires solving the homogeneous problem first, so we examine the homogeneous case first.

Consider

$$ay'' + by' + cy = 0$$

where a, b and c are given constants.

We have already seen how to solve this DE!

Idea 1 Suppose $y = e^{rt}$ is a solution to the DE. Then

$$ay'' + by' + cy = 0$$

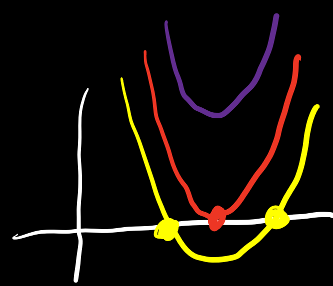
$$a(e^{rt})'' + b(e^{rt})' + ce^{rt} = 0$$

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$e^{rt}(ar^2 + br + c) = 0$$

Now, $e^{rt} \neq 0$, so

$$ar^2 + br + c = 0.$$



This is called the characteristic equation for the DE. The characteristic equation has two roots r_1 and r_2 which may be real and distinct, real and repeated, or complex conjugates depending on a, b, and c.

For now, let us assume r_1 and r_2 are distinct real roots. Then

$$y_1 = e^{r_1 t} \quad \text{and} \quad y_2 = e^{r_2 t}$$

both solve the DE.

Idea 2: If y_1 and y_2 are solutions of the DE, then $y = c_1 y_1 + c_2 y_2$ solves the DE.

$$\begin{aligned} ay'' + by' + cy &= a(c_1 y_1 + c_2 y_2)'' + b(c_1 y_1 + c_2 y_2)' \\ &\quad + c(c_1 y_1 + c_2 y_2) \\ &= a c_1 y_1'' + b c_1 y_1' + c c_1 y_1 \\ &\quad + a c_2 y_2'' + b c_2 y_2' + c c_2 y_2 \end{aligned}$$

$$= c_1(ar_1^2 + br_1 + c)e^{r_1 t}$$
$$+ c_2(ar_2^2 + br_2 + c)e^{r_2 t}$$
$$= 0$$

since r_1 and r_2 are roots of $ar^2 + br + c$.

The Initial Value Problem

We have our solution

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

and we can get a particular solution using the initial values

$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y_0'$$

to determine c_1 and c_2.

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is called the general solution.

Ex Solve the IVP

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}$$

First, the characteristic equation is

$$4r^2 - 8r + 3 = 0$$

$$(2r-1)(2r-3) = 0$$

$$r_1 = \frac{3}{2} \quad r_2 = \frac{1}{2}$$

Therefore, our general solution is

$$y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}$$

Now, we determine c_1 and c_2 using the ICs

$$2 = y(0) = c_1 + c_2$$

so $$c_1 = 2 - c_2 .$$

Now,

$$y = (2-c_2)e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}$$

and

$$y' = \frac{3}{2}(2-c_2)e^{\frac{3}{2}t} + \frac{1}{2}c_2 e^{\frac{1}{2}t}.$$

We have

$$\frac{1}{2} = y'(0) = \frac{3}{2}(2-c_2) + \frac{1}{2}c_2$$

$$1 = 3(2-c_2) + c_2$$

$$-5 = -2c_2$$

$$c_2 = \frac{5}{2}$$

Then

$$c_1 = 2 - c_2$$

$$= 2 - \frac{5}{2}$$

$$= -\frac{1}{2}.$$

Thus,

$$y = \frac{-1}{2} e^{\frac{3}{2}t} + \frac{5}{2} e^{\frac{1}{2}t} .$$

Long-time behavior

Our general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} .$$

What happens as $t \to \infty$?

§3.2 Solutions of Homogeneous Equations; Wronskian

Let p and q be continuous functions on an open interval I. The differential operator L is defined by

$$L[\phi] = \phi'' + p\phi' + q\phi$$

where ϕ is some function.

Note: L operates on ϕ, i.e. ϕ is the input to L

The value of L at a point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$

We will examine the second-order linear DE $L[\phi](t) = 0$ with ICs $y(t_0) = y_0$, $y'(t_0) = y_0'$.

We want to know whether the IVP

has a solution, more than one solution, and what is the structure and forms of solutions.

<u>Thm</u> (Existence and Uniqueness)

Let p, g, and q be continuous in an open interval I containing t_0. Then the IVP

$$y'' + p(t) y' + q(t) y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

has a unique solution $y = \varphi(t)$ in the interval I.

<u>Thm</u> (Principle of Superposition)

If y_1 and y_2 are solutions to the DE

$$L[y] = y'' + p(t) y' + q(t) = 0,$$

then the linear combination $c_1 y_1 + c_2 y_2$

is also a solution for any c_1 and c_2.

<u>Proof</u> Follows from the linearity of L.

Q: When can c_1 and c_2 be chosen to satisfy the ICs $y(t_0)=y_0$ and $y'(t_0)=y_0'$?

We need

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$

$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

in matrix form

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

When does this system have a unique solution?

Let $W(t_0) = \det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}$.

$W(t_0) \neq 0$ guarantees a unique solution $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

W is called the Wronskian of the solutions y_1 and y_2.

<u>Thm</u> Suppose y_1 and y_2 are two solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

It is possible to choose c_1 and

c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

Satisfies the DE and ICs $y(t_0) = y_0$, $y'(t_0) = y_0'$ if and only if the Wronskian

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2$$

is not zero at t_0.

Ex Consider the DE

$$y'' + 5y' + 6y = 0.$$

Find the Wronskian of y_1 and y_2.

The characteristic equation is

$$r^2 + 5r + 6 = 0$$

$$(r+3)(r+2) = 0$$

$r = -3 \quad r = -2$

so $y_1 = e^{-2t}$ and $y_2 = e^{-3t}$.

Now,

$$W[y_1, y_2] = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix}$$

$$= -3e^{-5t} + 2e^{-5t}$$

$$= -e^{-5t}$$

We see that $W \neq 0$ for all t, so any IC can be specified at any initial time t_0.

The expression

$$y = c_1 y_1(t) + c_2 y_2(t)$$

is called the general solution of $L[y]=0$.

y_1 and y_2 are said to form a fundamental set of solutions if and only if their Wronskian is nonzero for all t.

<u>Remark</u> Nonzero Wronskian $\Longleftrightarrow$ y_1 and y_2 are linearly indepent

<u>Ex</u> Show that $y_1 = t^{\frac{1}{2}}$ and $y_2 = \frac{1}{t}$ form a fundamental set of solutions of

$$2t^2 y'' + 3ty' - y = 0, \quad t>0.$$

1st we need to show y_1 and y_2 are solutions.

$$\begin{aligned} 2t^2 y_1'' + 3t y_1' - y_1 &= 2t^2 (t^{\frac{1}{2}})'' + 3t (t^{\frac{1}{2}})' - t^{\frac{1}{2}} \\ &= 2t^2 \left(\tfrac{1}{2} t^{-\frac{1}{2}}\right)' + 3t \tfrac{1}{2} t^{-\frac{1}{2}} - t^{\frac{1}{2}} \\ &= 2t^2 \cdot \tfrac{-1}{4} t^{-\frac{3}{2}} + \tfrac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}} \end{aligned}$$

$$= \frac{-1}{2} t^{\frac{1}{2}} + \frac{1}{2} t^{\frac{1}{2}}$$

$$= 0 \checkmark$$

so y_1 is a solution.

$$2t^2 y_2'' + 3t y_2' - y_2 = 2t^2 \left(\frac{1}{t}\right)'' + 3t \left(\frac{1}{t}\right)' - \frac{1}{t}$$

$$= 2t^2 \left(\frac{-1}{t^2}\right)' + 3t \frac{-1}{t^2} - \frac{1}{t}$$

$$= 2t^2 \frac{2}{t^3} + \frac{-3}{t} - \frac{1}{t}$$

$$= \frac{4}{t} - \frac{4}{t}$$

$$= 0 \checkmark$$

So y_2 is a solution.

Now, we calculate the Wronskian

$$W = \begin{vmatrix} t^{\frac{1}{2}} & \frac{1}{t} \\ \frac{1}{2} t^{\frac{-1}{2}} & \frac{-1}{t^2} \end{vmatrix}$$

$$= t^{\frac{1}{2}}(-t^{-2}) - \frac{1}{2} t^{-\frac{1}{2}} t^{-1}$$

$$= -t^{-3/2} - \frac{1}{2} t^{-3/2}$$

$$= \frac{-3}{2} t^{-3/2}$$

So $W \neq 0$ for $t > 0$. Thus, y_1 and y_2 form a fundamental set of solutions and the general solution is

$$y(t) = c_1 t^{\frac{1}{2}} + c_2 \frac{1}{t} .$$

<u>Thm</u> If $y = u(t) + i v(t)$ is a complex-valued solution of $L[\varphi] = 0$, then its real part u and imaginary part v are also solutions.

<u>Proof</u> Follows from linearity of L and

$L[y] = 0.$

Hint: If $a + ib = 0$, then $a = 0$ and $b = 0$.

Theorem (Abel's Theorem)

If y_1 and y_2 are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on I, then the Wronskian is

$$W[y_1, y_2](t) = c\, e^{-\int p(t)\,dt}$$

where c is a constant depending on y_1 and y_2 but not on t. W is either zero for all t in I or never zero in I.